Gauge Invariances in the Proca Model

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Abstract

We show that the abelian Proca model, which is gauge non-invariant with second class constraints can be converted into gauge theories with first class constraints. The method used, which we call Gauge Unfixing employs a projection operator defined in the original phase space. This operator can be constructed in more than one way, and so we get more than one gauge theory. Two such gauge theories are the Stückelberg theory, and the theory of Maxwell field interacting with an antisymmetric tensor field. We also show that the application of the projection operator does not affect the Lorentz invariance of this model.

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1. Introduction

Hamiltonian systems with second class constraints[1] have been the subject matter of interest for sometime now. Although their existence has been known for long these constraints were regarded as merely serving to reduce the degrees of freedom, and hence are removed by using the Dirac bracket formalism. First class constraints on the other hand imply the presence of gauge invariance.

Even though second class constraints by themselves do not imply gauge invariance in the corresponding systems, recent work[2,3] shows the possibility of underlying symmetries in such systems. These are revealed by converting the original second class system to equivalent theories which have gauge invariance. In the language of constraints this means the new theories will now have first class constraints.

Two methods are available for this conversion to equivalent gauge invariant theories. One is the BF method[2], which is basically formulated by extending the phase space of the original second class system. The other method is what we call Gauge Unfixing[3]; this, unlike the BF method is formulated within the original phase space itself. The important step in this is the construction of a certain projection operator which defines the gauge theory. For a second class system this operator is not unique. It can be constructed in more than one way and so we can have more than one gauge theory, all equivalent to the original second class system.

The advantages of treating a second class constrained system in this manner are obvious. The new gauge theory can be studied using well established techniques like BRST, Dirac quantisation, etc,. Further under gauge fixing the new theory goes back to the old (gauge non-invariant) one for a specific gauge. But other gauges can also be used, gauges which might yield physically relevant theories. We know from the results of Faddeev and Fradkin-Vilkovisky[4] that these gauges are all equivalent. Apart from this freedom in choosing the gauge, we also have the freedom in choosing the appropriate projection operator and thus the appropriate gauge theory.

In this paper we consider the abelian Proca model in the light of the above method. This model has only second class constraints. The projection operators are constructed.

For one choice of operator the resulting gauge theory has a trivial invariance, and the Hamiltonian is written entirely in terms of gauge invariant variables. The other choice for the projection operator gives a non-trivial gauge theory which will be shown to lead to the (gauge invariant) Stückelberg version[5] of the Proca model. Treated in a different manner, the Hamiltonian for this same non-trivial gauge theory leads to a model which has a massless antisymmetric tensor field interacting with the Maxwell field.

Many of these results have also been obtained by using the Batalin-Fradkin method[6,7] which, as we mentioned earlier, is formulated in an extended phase space. However we emphasize that our results are obtained through Gauge Unfixing, which involves no extension of the phase space. In other words the gauge theories that we obtain can be found *within* the phase space of the original second class (Proca) theory.

We also look at the Poincaré invariance of the new gauge theories. The Proca model that we start with has a manifestly Lorentz invariant Lagrangian. In phase space its Poincaré generators obey the Poincaré algebra through Dirac brackets[1]. We show that for either choice of the projection operator, these generators (even though they get modified by the projection operator) continue to obey the Poincaré algebra. The use of the projection operator thus does not affect Poincaré invariance.

In section 2 we introduce and summarize the gauge non-invariant Proca model. In section 3 we introduce the method of Gauge Unfixing and apply it to the Proca model. The two choices of the first class constraint are dealt with separately as cases (i) and (ii). Section 4 is devoted to conclusions. In the appendix we give the properties of the projection operator.

2. The Proca Model

The abelian Proca model is a $(3+1)$-dimensional theory given by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu, \tag{2.1}$$

with m the mass of the A_μ field. As usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $g_{\mu\nu} = diag(+,-,-,-)$.

In phase space, we have the momenta $\pi_\mu(x)$ conjugate to the $A^\mu(x)$ and the canonical Hamiltonian (after ignoring a total derivative term which arises in the Legendre transfor-

mation)

$$H_c = \int d^3x\, \mathcal{H}_c = \int d^3x \left(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}(A_0^2 - A_i^2) + A_0(\partial_i\pi_i) \right), \qquad (2.2)$$

with $\pi_i = -F_{0i}$. There are two second class constraints

$$\begin{aligned} Q_1 &= \pi_0(x) \approx 0, \\ Q_2 &= (-\partial_i\pi_i + m^2 A_0)(x) \approx 0, \end{aligned} \qquad (2.3)$$

where Q_1 is the primary constraint and Q_2 the secondary constraint. These two constraints together define the surface $\sum_2$ in the phase space. Their second class nature is seen by their non-zero Poisson brackets

$$\{Q_1(x), Q_2(y)\} = -m^2\,\delta(x-y). \qquad (2.4)$$

We thus have a 2×2 matrix E with elements $E_{ab} = \{Q_a(x), Q_b(y)\}$ $(a, b = 1, 2)$,

$$\begin{pmatrix} 0 & -m^2 \\ m^2 & 0 \end{pmatrix}, \qquad (2.5)$$

which has a non-zero determinant and hence an inverse E^{-1} everywhere in the phase space. The constraints (2.3) can be eliminated by replacing Poisson brackets (PBs) by Dirac brackets (DBs). For any two phase space functions B and C,

$$\begin{aligned} \{B(x), C(y)\}_{DB} &= \{B(x), C(y)\}_{PB} \\ &\quad - \int d^3u\, d^3v\, \{B(x), Q_a(u)\}_{PB} E^{-1}_{ab}(u,v) \{Q_b(v), C(y)\}_{PB}. \end{aligned} \qquad (2.6)$$

By construction the Dirac bracket of any variable with either of the constraints (2.3) is exactly zero. Further we have

$$\begin{aligned} \{A^i(x),\, \pi_j(y)\}_{DB} &= \delta^i_j \delta(x-y), \\ \{A^i(x),\, A^j(y)\}_{DB} &= \{\pi_i, \pi_j\}_{DB} = 0, \\ \{A_0(x),\, \pi_i(y)\}_{DB} &= 0, \\ \{A_0(x), A_i(y)\}_{DB} &= \frac{1}{m^2}\partial_{ix}\delta(x-y). \end{aligned} \qquad (2.7)$$

Thus the A^i and the π_j continue to remain canonical conjugate pairs. However from the last equation in (2.7), we see that A_0 is no longer independent of the π_i. This equation

is consistent with taking $Q_2 = 0$ as a strong equation[1] and replacing A_0 by $\frac{\vec{\nabla}\cdot\vec{\pi}}{m^2}$. Using $Q_2 = 0$, the canonical Hamiltonian (2.2) becomes

$$H_c = \int d^3x \left(\frac{1}{2}\pi_i\pi_i + \frac{m^2}{2}{A_i}^2 + \frac{F_{ij}F_{ij}}{4} + \frac{(\partial_i\pi_i)^2}{2m^2}\right). \tag{2.8}$$

The Lagrangian L in (2.1) is manifestly Lorentz invariant. To verify Poincaré invariance of the model in phase space, we look at the components of the energy-momentum and angular momentum tensors

$$\begin{aligned}
\mathcal{T}_{0\mu} &= -F_{0\alpha}(\partial_\mu A^\alpha) - g_{0\mu}\mathcal{L}, \\
\mathcal{M}_{0\mu\nu} &= x_\mu\mathcal{T}_{0\nu} - x_\nu\mathcal{T}_{0\mu} + \pi_\mu A_\nu - \pi_\nu A_\mu.
\end{aligned}$$

The Poincaré group generators $P_\mu = \int d^3x\ \mathcal{T}_{0\mu}$ and $M_{\mu\nu} = \int d^3x\ \mathcal{M}_{0\mu\nu}$ in phase space are then

$$\begin{aligned}
P_0 &= \int d^3x\ \left(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}[A_0^2 - A_i^2] + A_0(\partial_i\pi_i)\right), \\
P_i &= \int d^3x\ \left(\vec{\pi}\cdot(\partial_i\vec{A})\right), \\
M_{0i} &= \int d^3x\ \left(x_0\vec{\pi}\cdot(\partial_i\vec{A}) - x_i\mathcal{H}_c + \pi_0 A_i\right), \\
M_{ij} &= \int d^3x\ \left(x_i\,\vec{\pi}\cdot(\partial_j\vec{A}) - x_j\,\vec{\pi}\cdot(\partial_i\vec{A}) + \pi_i A_j - \pi_j A_i\right),
\end{aligned} \tag{2.9}$$

where in the first line, a total derivative term is ignored. Furthermore, though the term $-\pi_i A_0$ is present in $\mathcal{M}_{00i}$, it is absent in M_{0i}. This is so because the substitution (and rewriting) of the expression for $\mathcal{T}_{00}$ in M_{0i} gives rise to a term $+\pi_i A_0$ (apart from a total derivative) which cancels the $-\pi_i A_0$ already present. The resulting expression for M_{0i} is the one shown in (2.9), with $\mathcal{H}_c$ is the Hamiltonian density given in (2.2). Using the Dirac brackets (2.6) we find on the surface $\sum_2$,

$$\begin{aligned}
\{P_\mu, P_\nu\}_{DB} &= 0, \\
\{M_{\mu\nu}, P_\lambda\}_{DB} &= -g_{\mu\lambda}P_\nu + g_{\nu\lambda}P_\mu, \\
\{M_{\mu\nu}, M_{\sigma\rho}\}_{DB} &= -g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} + g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma}.
\end{aligned} \tag{2.10}$$

It is important to note (for later purposes) that the right hand sides of (2.10) (apart from total derivatives) also have terms involving the constraints (2.3), which have been put to zero here. The Poincaré algebra (2.10) thus confirms the Poincaré invariance of the Proca model in the Hamiltonian formulation.

3. Gauge Unfixing

We now derive the underlying symmetries of the Proca model using the gauge unfixing method[3]. For this we first note from (2.5) that each of the constraints in (2.3) is first class (i.e., has zero PB) with itself, but they are second class with respect to each other. Thus each is like a gauge fixing constraint to the other. Now if either of these constraints is retained and the other no longer considered a constraint, then we have a system with only a first class constraint. Accordingly we have two choices for our first class constraint. We consider these one by one.

Case (i)

We redefine the constraints (2.3) as

$$\begin{aligned} \chi(x) &= -\frac{1}{m^2} Q_1(x), \\ \psi(x) &= Q_2(x), \end{aligned} \tag{3.1}$$

so that, from (2.4) χ and ψ form a canonical conjugate pair. We now choose $\chi \cong 0$ as our first class constraint, and no longer consider $\psi \approx 0$. The dynamics will now be relevant on a new constrained surface $\sum_1$ defined by only $\chi \cong 0$ (the equality sign is changed from $\approx$ to $\cong$). In order that we have a gauge theory with transformations generated by χ, relevant physical quantities must be gauge invariant. In particular the Hamiltonian H_c of (2.2) is not gauge invariant, $\{\chi, H_c\} \ncong 0$ (on $\sum_1$). Hence to get gauge invariant observables, we define a projection operator

$$I\!\!P = \; : e^{-\int d^3x \; \psi \, \hat{\chi}} : , \tag{3.2}$$

where for any phase space functional B, we have $\hat{\chi} B \equiv \{\chi, B\}$. In applying (3.2) we adopt a particular ordering[3]; when $I\!\!P$ acts on any B, ψ should always be outside the Poisson bracket. We thus have the gauge invariant quantity $\widetilde{B}(x)$

$$\begin{aligned} \widetilde{B}(x) &= \; : e^{-\int d^3x \; \psi \, \hat{\chi}} : B(x) \\ &= B(x) - \int d^3y \; \psi(y) \{\chi(y), B(x)\} \\ &\quad + \frac{1}{2!} \int d^3y d^3z \; \psi(y)\psi(z) \{\chi(y), \{\chi(z), B(x)\}\} - .... + ... \end{aligned} \tag{3.3}$$

In particular, the gauge invariant Hamiltonian will be, using (2.2) and (3.3)

$$\begin{aligned}
\widetilde{H}_c &= H_c - \int d^3x\ \psi(x)(-\frac{1}{m^2})\psi(x) + \tfrac{1}{2}\int d^3x d^3y\ \psi(x)\psi(y)(\frac{-1}{m^2})\delta(x-y) \\
&= \int d^3x \left(\frac{\vec{\pi}^2}{2} + A_0\vec{\nabla}\cdot\vec{\pi} + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}(A_0^2 - \vec{A}^2) + \frac{\psi^2}{2m^2}\right).
\end{aligned} \tag{3.4}$$

It can be checked that $\{\chi(x), \widetilde{H}_c\} = 0$. Thus $\chi \cong 0$ and $\widetilde{H}_c$ describe a consistent gauge theory. This gauge theory goes back to the original Proca model when we consider and use $\psi \approx 0$ as the gauge fixing condition.

The χ is the generator of gauge transformations. It can be checked that the A^i and the π_i $(i = 1, 2, 3)$ are all gauge invariant. However A_0 is not, since $A_0 \to A_0' = A_0 + \lambda$, for infinitesimal gauge transformations. Here λ is the transformation parameter.

The Hamiltonian $\widetilde{H}_c$ though gauge invariant, involves gauge non-invariant fields. Using the explicit form $\psi = \left(-\vec{\nabla}\cdot\vec{\pi} + m^2 A_0\right)$, it can be rewritten in terms of only gauge invariant fields,

$$\widetilde{H}_c = \int d^3x \left(\frac{\pi_i^2}{2} + \frac{m^2 A_i^2}{2} + \frac{F_{ij}F_{ij}}{4} + \frac{(\partial_i\pi_i)^2}{2m^2}\right), \tag{3.5}$$

where the fields $A^i(x)$ and $\pi_j(x)$ continue to form canonical conjugate pairs. Note that $\widetilde{H}_c$ in (3.5) is just the Dirac bracket Hamiltonian (2.8) of the original Proca theory.

We now look at the Poincaré invariance of the new gauge theory. In order that the Poincaré group generators be physical observables, they must be gauge invariant with respect to χ. To obtain these, we first apply $I\!P$ on the quantities P_μ, $M_{\mu\nu}$ of (2.9),

$$\begin{aligned}
I\!P(P_0) &= \widetilde{P}_0 \ = P_0 + \int d^3x \left(\frac{\psi^2}{2m^2}\right), \\
I\!P(P_i) &= \widetilde{P}_i \ = P_i, \\
I\!P(M_{0i}) &= \widetilde{M}_{0i} = M_{0i} - \int d^3x \left(\frac{x_i\psi^2}{2m^2}\right) \\
I\!P(M_{ij}) &= \widetilde{M}_{ij} = M_{ij}.
\end{aligned} \tag{3.6}$$

These projected quantities can be verified to be gauge invariant. Thus in order that they be gauge invariant, P_0 and M_{0i} get modified. The Poincare algebra is verified by looking at the Poisson brackets of the projected quantities (3.6). To this end we use certain properties of the projection operator (see appendix). Using (A.6), the Dirac brackets (2.10), (A.5)

and (A.4), we find on the surface $\sum_1 (\chi \cong 0)$

$$
\begin{aligned}
\{\widetilde{P}_\mu, \widetilde{P}_\nu\} &\cong 0, \\
\{\widetilde{M}_{\mu\nu}, \widetilde{P}_\lambda\} &\cong -g_{\mu\lambda}\widetilde{P}_\nu + g_{\nu\lambda}\widetilde{P}_\mu, \\
\{\widetilde{M}_{\mu\nu}, \widetilde{M}_{\sigma\rho}\} &\cong -g_{\mu\sigma}\widetilde{M}_{\nu\rho} + g_{\nu\sigma}\widetilde{M}_{\mu\rho} + g_{\mu\rho}\widetilde{M}_{\nu\sigma} - g_{\nu\rho}\widetilde{M}_{\mu\sigma},
\end{aligned}
\tag{3.7}
$$

which shows that the Poincaré algebra is not affected by the projection operator $I\!\!P$ (3.2). In this context it must be noted that it is necessary here to have $I\!\!P-$projected Poincaré generators instead of the old ones (2.9). If we consider the old generators (2.9), then their PB or DB algebra (2.10) will in general involve both χ and ψ, which can both be put to zero (surface $\sum_2$) *only in* the original second class theory. In our new gauge theory, only χ can be put to zero (on $\sum_1$), and so the old generators (2.9) no longer give the Poincaré algebra. But if instead the $I\!\!P-$projected quantities (3.6) are used, even if their Poisson brackets give extra terms involving ψ, these get eliminated due to the property (A.4), and Poincaré algebra is obtained.

The inverse Legendre transformation for the Hamiltonian $\widetilde{H}_c$ (3.4) will result in a Lagrangian which is not manifestly Lorentz invariant. We do not consider this here.

Case (ii)

To consider a different choice of first class constraint, we reclassify the constraints (2.3) as

$$
\begin{aligned}
\chi'(x) &= \frac{1}{m^2}Q_2(x) = \frac{1}{m^2}\left(-\vec{\nabla}\cdot\vec{\pi} + m^2 A_0\right) \\
\psi'(x) &= Q_1(x) = \pi_0(x),
\end{aligned}
\tag{3.8}
$$

which, as in the earlier classification (3.1), form canonical conjugate pairs. We choose $\chi' \cong' 0$ (note the change in equality sign) to be our first class constraint, and disregard $\psi' \approx 0$. Then $\chi' \cong' 0$ will define a new constrained surface $\sum_1'$, different from the earlier $\sum_2$ and $\sum_1$. Our new gauge theory is now to be defined on this new $\sum_1'$.

As in case (i), we must have observables gauge invariant under gauge transformations generated here by χ'. Quantities like the second class Hamiltonian H_c of (2.2) do not in general satisfy this requirement. Further the Hamiltonian $\widetilde{H}_c$ of (3.4), which was gauge invariant in the earlier case (i) is not so here, $\{\chi', \widetilde{H}_c\} \ncong' 0$. Hence we define and construct

a new projection operator

$$
\begin{aligned}
I\!\!P' &= :e^{-\int d^3x\ \psi'(x)\hat{\chi}'(x)}: \\
I\!\!P'(B) &\equiv \widetilde{B}' \\
\hat{\chi}'B &= \{\chi', B\},
\end{aligned}
\tag{3.9}
$$

where B is any phase space functional. Again, as in (3.2), we have here a particular ordering — the ψ' is always outside the PBs occuring in the series expansion of $I\!\!P'(B)$.

It must be noted that the $I\!\!P'$ and $\hat{\chi}'$ in (3.9) are *not* the same as the $I\!\!P$ and $\hat{\chi}$ in (3.2). Thus the gauge theory defined by $I\!\!P'$ is in general different from the one defined by $I\!\!P$. For instance, the gauge invariant Hamiltonian $I\!\!P'(H_c)(= \widetilde{H}_c^{'})$ here is different from the $\widetilde{H}_c$ of eqn.(3.4),

$$
\begin{aligned}
\widetilde{H}_c^{'} &= H_c - \int d^3x\ \left[\psi'(\vec{\nabla}\cdot\vec{A}) + \frac{1}{2m^2}\psi'\vec{\nabla}^2\psi'\right] \\
&= \int d^3x \left[\frac{\pi_i^2}{2} + A_0(\partial_i\pi_i) + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}(A_0^2 - A_i^2)\right. \\
&\qquad \left. -\pi_0(\vec{\nabla}\cdot\vec{A}) + \tfrac{1}{2m^2}(\vec{\nabla}\pi_0)^2\right].
\end{aligned}
\tag{3.10}
$$

It can be verified that $\{\chi'(x), \widetilde{H}_c^{'}\} = 0$. Thus $\chi' \cong' 0$ and $\widetilde{H}_c^{'}$ define our new gauge theory. This goes back to the Proca theory under the gauge condition $\psi' \approx 0$. The Hamiltonian $\widetilde{H}_c^{'}$ goes back to the second class Hamiltonian (2.2).

The gauge transformations are generated by χ', and unlike in case (i) (where A^i and π_j were gauge invariant), here the gauge invariant fields are A_0 and π_i. As for the remaining fields, we have, for a transformation parameter $\mu(x)$

$$
\begin{aligned}
A_i \rightarrow A_i' &= A_i + \int d^3x\ \mu(x)\{A_i, \chi'(x)\} \\
&= A_i - \frac{1}{m^2}(\partial_i\mu), \\
\pi_0 \rightarrow \pi_0' &= \pi_0 - \mu.
\end{aligned}
\tag{3.11}
$$

It can also be verified explicitly using (3.11) that $\widetilde{H}_c^{'}$ is gauge invariant.

Before we look further at the Hamiltonian $\widetilde{H}_c^{'}$, we look for Poincaré invariance in this new gauge theory. As in case (i), the group generators must be gauge invariant, this time with respect to χ'. It can be seen that neither the quantities (2.9) of the second class Proca theory, nor the quantities (3.6) have zero PBs with χ'. Hence we apply the operator $I\!\!P'$

(3.9) on all the quantities $P_\mu, M_{\mu\nu}$ of (2.9). Noting from (3.9) that the operation of $I\!P'$ results in a series, we get the gauge invariant quantities

$$
\begin{aligned}
\widetilde{P}_0{}' &= \int d^3x \left(\frac{\pi_i^2}{2} + A_0(\partial_i\pi_i) - \frac{m^2}{2}(A_0^2 - A_i{}^2) + \frac{1}{4}F_{ij}^2 \right. \\
&\qquad\qquad \left. - \pi_0(\vec{\nabla}\cdot\vec{A}) + \frac{1}{2m^2}(\vec{\nabla}\pi_0)^2 \right), \\
\widetilde{P}_i{}' &= P_i - \int d^3x \; \frac{1}{m^2}\psi'[-\partial_i(\vec{\nabla}\cdot\vec{\pi})] \\
&= \int d^3x \, (\pi_\mu \partial_i A^\mu - \psi'\partial_i\chi'), \\
\widetilde{M}_{0i}' &= M_{0i} + \int d^3x \left(\frac{x_0}{m^2}\pi_0\partial_i(\vec{\nabla}\cdot\vec{\pi}) + x_i\pi_0(\vec{\nabla}\cdot\vec{A}) + \frac{x_i}{2m^2}\pi_0\vec{\nabla}^2\pi_0 \right) \qquad (3.12) \\
&= \int d^3x \left[x_0\pi_\mu\partial_i A^\mu - x_0\psi'\partial_i\chi' - x_i\widetilde{\mathcal{H}}_c' + \pi_0 A_i \right], \\
\widetilde{M}_{ij}' &= M_{ij} + \int d^3x \; \psi'(x_i\partial_j - x_j\partial_i)(\frac{\vec{\nabla}\cdot\vec{\pi}}{m^2}) \\
&= \int d^3x \, (x_i\,\pi_\mu\partial_j A^\mu - x_j\,\pi_\mu\partial_i A^\mu + \pi_i A_j - \pi_j A_i \\
&\qquad\qquad - \psi'(x_i\partial_j - x_j\partial_i)\chi'),
\end{aligned}
$$

where we have used (2.9), (2.3), (3.8) and the Hamiltonian density $\widetilde{\mathcal{H}}_c'$ of (3.10). Also we have ignored total derivative terms in writing the expressions for $\widetilde{P}_0'$ and $\widetilde{M}_{0i}'$. It can be verified that the above projected quantities are indeed gauge invariant with respect to χ'.

We now verify the Poincaré algebra. The old generators $P_\mu, M_{\mu\nu}$ of (2.9) will not serve this purpose here. This is because, as mentioned in section 2, the Dirac brackets (2.10) will in general involve extra terms involving both the χ' and ψ'. In the present gauge theory, only the χ' can be put to zero (surface $\sum_1'$) and *not* the ψ', in which case we will not have the Poincaré algebra.

For similar reasons the $\widetilde{P}_\mu$ and $\widetilde{M}_{\mu\nu}$ of (3.6) which obeyed the Poincaré algebra in the gauge theory of case(i), cannot do so here. The PBs (3.7) among $\widetilde{P}_\mu$ and $\widetilde{M}_{\mu\nu}$ involved extra terms in $\pi_0(=\psi')$, which cannot be put to zero here. Consequently we are left with verifying if the $\widetilde{P}_\mu'$, $\widetilde{M}_{\mu\nu}'$ of (3.12) satisfy the Poincaré algebra.

As in case (i), we use the properties of the projection operator given in the appendix. We use the Dirac brackets (2.10), and using (A.6), (A.5) and (A.4) we eliminate the extra

terms in ψ'. We thus get on the constraint surface $\sum_1'$

$$
\begin{aligned}
\{\widetilde{P}_\mu^{'}, \widetilde{P}_\nu^{'}\} &= I\!\!P'\left(\{P_\mu, P_\nu\}_{DB}\right) \ \cong' 0, \\
\{\widetilde{M}_{\mu\nu}^{'}, \widetilde{P}_\lambda^{'}\} &= I\!\!P'\left(\{M_{\mu\nu}, P_\lambda\}_{DB}\right) \quad \cong' -g_{\mu\lambda}\widetilde{P}_\nu^{'} + g_{\nu\lambda}\widetilde{P}_\mu^{'}, \\
\{\widetilde{M}_{\mu\nu}^{'}, \widetilde{M}_{\sigma\rho}^{'}\} &= I\!\!P'\left(\{M_{\mu\nu}, M_{\sigma\rho}\}_{DB}\right) \\
&= -g_{\mu\sigma}\widetilde{M}_{\nu\rho}^{'} + g_{\nu\sigma}\widetilde{M}_{\mu\rho}^{'} + g_{\mu\rho}\widetilde{M}_{\nu\sigma}^{'} - g_{\nu\rho}\widetilde{M}_{\mu\sigma}^{'}.
\end{aligned}
\tag{3.13}
$$

Thus the Poincaré algebra is satisfied and $\widetilde{P}_\mu^{'}$, $\widetilde{M}_{\mu\nu}^{'}$ are the generators of this group in the gauge theory defined by $\chi' \cong' 0$. The application of $I\!\!P'$ thus does not affect the Poincaré invariance of the Proca model.

We now return to the gauge invariant Hamiltonian $\widetilde{H}_c^{'}$ of (3.10). The equations of motion are

$$
\begin{aligned}
\dot{A}_0 &= -\vec{\nabla}\cdot\vec{A} - \frac{\vec{\nabla}^2\psi'}{m^2}, \\
\dot{\pi}_0 &= Q_2 = -m^2\chi' \quad \cong' 0, \\
\dot{A}_i &= -\pi_i + \partial_i A_0, \\
\dot{\pi}_i &= \partial_j F_{ji} + m^2 A_i - \partial_i\psi'.
\end{aligned}
\tag{3.14}
$$

We once again see the equivalence of this gauge theory (case(ii)) with the original Proca model. Under the gauge fixing condition $\psi' \approx 0$, the equations (3.14) go back to the equations of motion for the Proca model.

We now consider the passage from $\widetilde{H}_c^{'}$ to the Lagrangian formulation. Using $\dot{\pi}_0 = Q_2$ from (3.14), we rewrite $\widetilde{H}_c^{'}$ as

$$
\begin{aligned}
\widetilde{H}_c^{'} &= \int d^3x \left(\frac{\vec{\pi}^2}{2} + \left[A_0 - \frac{1}{m^2}\partial_0\pi_0\right](\vec{\nabla}\cdot\vec{\pi}) + \frac{1}{4}F_{ij}F_{ij} - \pi_0\,\vec{\nabla}\cdot\vec{A}\right. \\
&\qquad \left. -\frac{m^2}{2}\left[(A_0 - \tfrac{1}{m^2}\partial_0\pi_0)^2 - \vec{A}^2\right] - \frac{1}{2m^2}\left[(\partial_0\pi_0)^2 - (\vec{\nabla}\pi_0)^2\right]\right) \\
&= \int d^3x \left(\frac{\vec{\pi}^2}{2} + A_0'\vec{\nabla}\cdot\vec{\pi} + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}(A_0'^{\,2} - \vec{A}^2)\right. \\
&\qquad \left. -\pi_0\vec{\nabla}\cdot\vec{A} - \frac{1}{2m^2}(\partial_\mu\pi_0)^2)\right),
\end{aligned}
\tag{3.15}
$$

where we have called $(A_0 - \frac{1}{m^2}\partial_0\pi_0) = A_0'$. The equation of motion for A_0' is the same as for A_0, because of $\dot{\pi}_0 = m^2\chi'(\cong' 0)$. Further in eqn.(3.14) for $\dot{A}_i$, $\partial_i A_0$ becomes $\partial_i A_0'$. Using (3.15) and ignoring the prime on A_0', we write the Lagrangian $L = \int d^3x \left(\pi_0\dot{A}_0 + \pi_i\dot{A}^i - \widetilde{\mathcal{H}}_c^{'}\right)$

as

$$
\begin{aligned}
L &= \int d^3x \left(\pi_0 \partial_\mu A^\mu - \pi_i(-\pi_i + \partial_i A_0) - \frac{\pi_i^2}{2} - \frac{1}{4} F_{ij} F_{ij} \right. \\
&\qquad \left. - A_0(\partial_i \pi_i) + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2m^2} (\partial_\mu \pi_0)^2 \right) \\
&= \int d^3x \left(\pi_0 \partial_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2m^2} (\partial_\mu \pi_0)^2 \right).
\end{aligned} \tag{3.16}
$$

If we now consider the π_0 (or ψ') to be a new field appearing in the Lagrangian, and then rescale π_0 to $\theta = \frac{-1}{m^2}\pi_0$, we can rewrite L as

$$
\begin{aligned}
L &= \int d^3x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu - m^2 \theta \partial_\mu A^\mu + \frac{m^2}{2} (\partial_\mu \theta)^2 \right) \\
&= \int d^3x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta) \right).
\end{aligned} \tag{3.17}
$$

We have ignored a total derivative term in the second line in (3.17). We thus arrive at the Stückelberg Lagrangian[5]. The θ field is identified with the so-called Stückelberg scalar, whose gauge transformation cancels that of the A_μ field, thus making L invariant. Note that this L looks like the Proca Lagrangian, but here in (3.17) the fields $(A_\mu + \partial_\mu \theta)$ are all gauge invariant. It may also be noted that the Stückelberg Lagrangian in (3.17) goes back to the Proca Lagrangian (2.1) under the (unitary) gauge condition $\theta = 0$.

A remark is in order at this stage. Using the Batalin-Fradkin method a similar result has been obtained by Banerjee et al, Sawayanagi[6] and Kim et al[7]. There the phase space is enlarged by introducing an extra canonical conjugate pair of fields. The extra field is identified with the Stückelberg scalar, and additional terms in this extra field appear in the Hamiltonian to make it gauge invariant.

In contrast, we have found the Stückelberg scalar *within* the original phase space itself. This is just the $\psi'(= \pi_0)$ of (3.8). As we have shown, gauge unfixing *does not* allow this ψ' to be put to zero. As a result extra terms in ψ' appear in the gauge invariant Hamiltonian. These extra terms correspond to the additional terms appearing in the BF gauge invariant Hamiltonian[6,7]. Thus the ψ' (with rescaling) of the gauge unfixing method is just the extra field introduced in the BF method[6,7]. Indeed this identification is confirmed when we go back to the second class Proca model. In the gauge unfixing method this is achieved by gauge fixing with $\psi' \approx 0$, whereas in the BF method the extra field is put to zero.

We also mention that the Stückelberg Lagrangian is manifestly Lorentz invariant, thus confirming the Poincaré algebra (3.13) that we obtained using modified Poincaré group generators.

The gauge theory of case (ii) can be related to another model too. To see this, we rewrite the Hamiltonian $\widetilde{H}_c^{'}$ of (3.10) as

$$\widetilde{H}_c^{'} = \int d^3x \, \left(\frac{\vec{\pi}^2}{2} + \frac{F_{ij}F_{ij}}{4} + F_0 \, \vec{\nabla} \cdot \vec{\pi} - \frac{m^2}{2} F_0^2 + \frac{m^2}{2} F_i^2 \right), \tag{3.18}$$

where

$$\begin{aligned} F_i &= A_i - \frac{\partial_i \pi_0}{m^2}, \\ F_0 &= A_0. \end{aligned} \tag{3.19}$$

Using (3.11), we see that F_0 and F_i are gauge invariant fields. Thus the Hamiltonian $\widetilde{H}_c^{'}$ in (3.10) involves gauge non-invariant fields, whereas the $\widetilde{H}_c^{'}$ in (3.18) has only gauge invariant fields. In contrast to the A_0 and A_i having zero Poisson brackets among themselves, we have here

$$\begin{aligned} \{F_0, F_0\} &= \{F_i, F_j\} = 0, \\ \{F_0(x), F_i(y)\} &= \frac{1}{m^2} \, \partial_{ix} \delta(x-y). \end{aligned} \tag{3.20}$$

Thus the price one pays for considering gauge invariant fields is the non-zero PB in (3.20). Note that the above PBs among F_0, F_i are just the Dirac brackets (2.7) among the A_0, A_i fields in the original Proca system. We next define $G_{\mu\nu} = \partial_\mu F_\nu - \partial_\nu F_\mu$, and find that

$$\partial_\mu G^{\mu\nu} = -m^2 F^\nu + g^{0\nu} \, \vec{\nabla}^2 \chi'. \tag{3.21}$$

Thus modulo a term in χ', eqn.(3.21) is similar to the corresponding equation in the Proca model, $\partial_\mu F^{\mu\nu} = -m^2 A^\nu$ which however involves gauge non-invariant fields.

Since (3.21) leads to $\partial_\mu F^\mu = 0$ the F_μ fields can be written in terms of a gauge invariant antisymmetric tensor field $A^{\mu\nu}$

$$\begin{aligned} F_\mu &= \tfrac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \partial^\nu A^{\lambda\sigma} \\ &= \tfrac{1}{6} \epsilon_{\mu\nu\lambda\sigma} G^{\nu\lambda\sigma}, \end{aligned} \tag{3.22}$$

where we have the totally antisymmetric quantity $G^{\nu\lambda\sigma} = \partial^{[\nu} B^{\lambda\sigma]}$. The Hamiltonian $\widetilde{H}_c^{'}$

now becomes

$$
\begin{aligned}
\widetilde{H}_c^{'} &= \int d^3x \left(\frac{\pi_i^2}{2} + \frac{F_{ij}F_{ij}}{4} + F_0(\partial_i \pi_i) - \frac{m^2}{8}(\epsilon_{ijk}\partial^i A^{jk})^2 + \frac{m^2}{8}(\epsilon_{i\nu\lambda\sigma}\partial^\nu A^{\lambda\sigma})^2 \right) \\
&= \int d^3x \left(\frac{\pi_i^2}{2} + \frac{F_{ij}F_{ij}}{4} + F_0(\partial_i \pi_i) + \frac{m^2}{12} G_{ijk}G^{ijk} + \frac{m^2}{4} G_{0jk}G^{0jk} \right)
\end{aligned}
\tag{3.23}
$$

with $\epsilon_{ijk} = \epsilon_{0ijk}$.Note that (3.23) involves only gauge invariant fields.

It is more interesting to consider gauge non-invariant antisymmetric tensor fields. Recall that the gauge invariant Hamiltonian $\widetilde{H}_c^{'}$ was first obtained as a series (3.10) in the π_0, which was later redefined to be the Stückelberg scalar $\theta (= \frac{-\pi_0}{m^2})$. Instead of a scalar field, we can introduce a tensor field, while still retaining gauge invariance. For this, we use (3.19) and (3.22) to write

$$
\begin{aligned}
\partial_i \pi_0 &= m^2 (A_i - \tfrac{1}{2}\epsilon_{i\mu\nu\lambda}\partial^\mu A^{\nu\lambda}) \\
&= \tfrac{1}{2}\epsilon_{ijk}\pi^{jk}, \\
\pi_{jk} &= m^2(\epsilon_{jkm}A^m + G_{0jk})
\end{aligned}
\tag{3.24}
$$

The Hamiltonian of (3.10) now becomes

$$
\widetilde{H}_c^{'} = H_c + \int d^3x \left(\frac{1}{4m^2}\pi_{ij}\pi^{ij} + \frac{1}{2}\epsilon_{ijk}A^i\pi^{jk} \right) \tag{3.25}
$$

where H_c is the Proca Hamiltonian (2.2). Thus in place of a (finite) series in a scalar field, we now have $\widetilde{H}_c^{'}$ to be a series in the tensor field π_{ij}. The gauge theory involving θ (or π_0) had the A_μ field interacting with the θ field; here A_μ interacts with an antisymmetric tensor field. Note that the unitary gauge $\pi_{ij} = 0$ takes $\widetilde{H}_c^{'}$ back to the Proca Hamiltonian H_c; this, from (3.24) is just the $\pi_0 = 0$ used earlier.

The Hamiltonian $\widetilde{H}_c^{'}$ in (3.25) is invariant under gauge transformations generated by $\left(\frac{-\vec{\nabla}\cdot\vec{\pi}}{m^2} + \frac{1}{2}\epsilon_{ijk}\partial^i A^{jk} \right)$, which is obtained from $\chi' = \frac{1}{m^2}(-\partial_i \pi_i + m^2 A_0)$ using (3.22). The fields $A^{\mu\nu}$ are gauge invariant, from (3.22). The tensor π_{jk} however is not; using

$$
\{A^{ij}(x), \pi_{mn}(y)\} = (\delta^i_m \delta^j_n - \delta^i_n \delta^j_m)\, \delta(x-y), \tag{3.26}
$$

we find the variation $\pi^{jk} \to \pi^{jk} - \epsilon^{ijk}\partial_i \mu$, where μ is the transformation parameter. Note that the relation (3.26) and the above variation of π_{jk} are consistent with (3.24) and the variation (3.11) of π_0.

The Hamiltonian (3.25) is very similar to the one obtained by Sawayanagi[6], who has used the BF method. The extra fields introduced were just the tensor field π_{ij} and $(\frac{1}{2}\epsilon_{ijk}\partial^i A^{jk} - A_0)$, which were used to write down the gauge invariant Hamiltonian as a (finite) series (this Hamiltonian[6] has an extra term involving $(\frac{1}{2}\epsilon_{ijk}\partial^i A^{jk} - A_0)$, which is zero in our case, see (3.22)). Our result (3.25) however is obtained *within* the original phase space.

The Hamiltonian $\widetilde{H}_c^{'}$ may not lead to a manifestly Lorentz invariant Lagrangian involving the Maxwell and tensor fields (we have not considered Lorentz invariance in phase space here, since it has already been verified in (3.13)). We can however write down such a Lagrangian[6,8] which gives the Hamiltonian (3.25),

$$\widetilde{L}^{'} = \int d^3x \left(-\frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{6}\epsilon_{\mu\nu\rho\sigma}A^\mu G^{\nu\rho\sigma} + \frac{m^2}{12}G_{\mu\nu\rho}G^{\mu\nu\rho}\right), \tag{3.27}$$

with $G_{\mu\nu\alpha}$ antisymmetrised in all the indices. The phase space involves $A^\mu, A^{\mu\nu}$ and the canonical momenta

$$\begin{aligned} \pi_i &= -F_{0i} & \qquad \pi_{ij} &= m^2(\epsilon_{ijk}A^k + G_{0ij}) \\ \pi_0 &= 0 & \qquad \pi_{0i} &= 0 \end{aligned} \tag{3.28}$$

with the second line giving the primary constraints. The canonical Hamiltonian is

$$\begin{aligned} H_{inv} = \int d^3x \, \Big(& \frac{1}{2}\pi_i^2 + \frac{m^2}{2}A_i^2 + \frac{1}{4}F_{ij}^2 + \frac{1}{4m^2}\pi_{ij}^2 + \frac{1}{2}\epsilon_{ijk}A_i\pi_{jk} \\ & + \frac{m^2}{12}G_{ijk}^2 - A_0(-\partial_i\pi_i + \frac{m^2}{2}\epsilon_{ijk}\partial^i A^{jk}) + A_{0j}\partial_i\pi_{ij}\Big). \end{aligned} \tag{3.29}$$

The time independence of the primary constraints in (3.28) yield the secondary constraints,

$$\begin{aligned} -\vec{\nabla}\cdot\vec{\pi} + \frac{m^2}{2}\epsilon_{ijk}\partial^i A^{jk} &= 0, \\ \partial_i\pi_{ij} &= 0. \end{aligned} \tag{3.30}$$

Modulo these constraints and using (3.22), we find that (3.29) is just the Hamiltonian $\widetilde{H}_c^{'}$ of (3.25). Note that the constraints in (3.28) and (3.30) are all first class, showing that (3.27) describes a gauge theory.

4. Conclusion

In this paper we have revealed gauge symmetries inherently present in the gauge non-invariant Proca model. We have used the Gauge Unfixing method, the central object of

which is the projection operator. We have shown that this operator defines the gauge theory by projecting all relevant quantities (constructed initially on the second class constrained surface) onto a first class constrained surface. This projection operator is not unique; there are two different operators, which implies two different gauge theories. We have shown that one of these results in a trivial gauge invariance, and the other gives a non-trivial one. In each of these gauge theories we have verified Poincaré invariance by (necessarily) modifying the Poincaré generators of the original Proca model.

For the first gauge theory (case[i]), the corresponding Lagrangian is not manifestly Lorentz invariant (even though in phase space Lorentz invariance is confirmed). As for the second gauge theory the passage to the Lagrangian formulation results in a manifestly Lorentz invariant Lagrangian, thus confirming the Lorentz invariance shown in phase space. Further this Lagrangian is just the Stückelberg Lagrangian, which was proposed quite sometime back by Stückelberg[5] by adding extra terms in an extra (Stückelberg) field directly to the Proca Lagrangian. From the constraints point of view our method is thus consistent with the Stückelberg formulation.

The Stückelberg Lagrangian has also been derived using the Batalin-Fradkin (BF) method[6,7], which is formulated by enlarging the phase space. We emphasize that the Gauge Unfixing method derives this Lagrangian *without* any extension of the phase space (similar conclusions have been arrived at for other systems also — the abelian Chern-Simons theory and the abelian chiral Schwinger model[3]). Thus we have a connection between the two methods.

We have also shown that the gauge theory of case(ii) leads to another formulation, that of the Maxwell field interacting with an antisymmetric tensor field. Whereas this was shown by Banerjee and Sawayanagi[6] to arise in an extended phase space, our analysis here shows that the original phase space is sufficient to reproduce such a theory.

It would be interesting to see how well the method works for the non-abelian Proca model. In this model, it is not just the $\pi_0 \approx 0$ constraints and the Gauss law constraints which are second class with each other, but the Gauss law constraints are second class among themselves. It may be possible to use the Gauge Unfixing method (under certain conditions, see [3]) for these systems too. Work is in progress in this direction.

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Note Added After completion of this work, we became aware of another paper[9], where the conversion of the abelian Proca model into a first class system has been discussed.

APPENDIX

The projection operator $I\!\!P =: e^{-\int d^3x\ \psi\hat{\chi}}$: has the following properties:

$$I\!\!P^2 \cong I\!\!P \tag{A.1}$$

$$I\!\!P(bB + c\,C) = b\widetilde{B} + c\,\widetilde{C} \tag{A.2}$$

$$\hat{\chi} I\!\!P \cong 0 \tag{A.3}$$

$$I\!\!P(\psi) = \widetilde{\psi} \cong 0 \tag{A.4}$$

$$(\widetilde{BC}) = \widetilde{B}\widetilde{C} \tag{A.5}$$

$$\{\widetilde{B}, \widetilde{C}\} \cong I\!\!P\,(\{B, C\}_{DB}) \tag{A.6}$$

$$\{\widetilde{B}, \{\widetilde{C}, \widetilde{D}\}\} + \{\widetilde{C}, \{\widetilde{D}, \widetilde{B}\}\} + \{\widetilde{D}, \{\widetilde{B}, \widetilde{C}\}\} \cong 0 \tag{A.7}$$

where the symbol $\cong$ implies equality on the surface defined by only $\chi \cong 0$. The proofs for the above properties can be found in [3].

References

(1) P A M Dirac, *Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University, New York, 1964*;
A J Hanson, T Regge and C Teitelboim, *Accademia Nazionale dee Lincei, Rome, 1976.*

(2) I A Batalin and E S Fradkin, *Phys. Lett.* **B 180 (1986) 157; Nucl. Phys. B 279 (1987) 514**;
I A Batalin and I V Tyutin, *Int. J. Mod. Phys.* **A 6 (1991), 3255 and references therein**.

(3) P Mitra and R Rajaraman *Ann. Phys. (N Y)* **203 (1990) 137, 157**;
R Anishetty and A S Vytheeswaran *J. Phys.* **A 26 (1993) 5613**;
A S Vytheeswaran *Ann. Phys. (N Y)* **236 (1994) 297**.

(4) L D Faddeev *Theor. Math. Phys.* **1 (1969) 1**;
E S Fradkin and G A Vilkovisky *Phys. Lett.* **B 55 (1975) 224**;
M Henneaux *Phys. Rep.* **126 (1981) 1**.

(5) E C G Stückelberg *Helv. Phys. Act.* **30 (1957) 209.**

(6) N Banerjee and R Banerjee *Mod Phys Lett.* **A11 (1996) 1919**;
H Sawayanagi *Mod. Phys. Lett.* **A 10 (1995) 813.**

(7) Y W Kim, M I Park, Y J Park, S J Yoon *hep-th/9512110; To appear in International Journal of Modern Physics A.*

(8) T J Allen, M J Bowick and A Lahiri *Mod. Phys. Lett.* **A 6 (1991) 559;**
A Lahiri *Mod. Phys. Lett.* **A 8 (1993) 2403.**

(9) Omar F Dayi, *Phys. Lett.* **B 210 (1988) 147,**
C Bizdadea and S O Saliu *EuroPhys. Lett.* **32 (1995) 307-312.**